A STATISTICAL THEORY FOR PREDICTING RESPONSE OF MATERIALS THAT POSSESS A DISORDERED STRUCTURE

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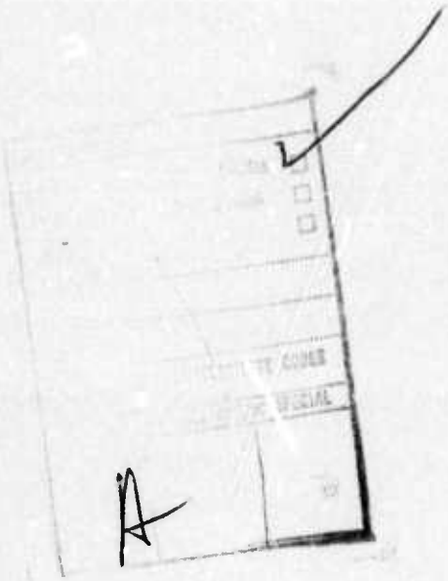

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ARMY MATERIALS AND MECHANICS RESEARCH CENTER

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ABSTRACT

A statistical continuum is one with material properties that can be described only in probabilistic terms. Such continua are encountered in an ever increasing number of important engineering and scientific problems. Examples include the response of heterogeneous solids, the mechanics of the flow of blood, the dispersion of additives by a turbulent fluid, the scattering of waves by turbulence, by a temperature substructure, etc. In this report we present a foundation that can be used to describe all of these physically different phenomena. Emphasis is placed on the unique difficulties that present themselves to both the theoretical and the experimentalist and the progress that has been made in surmounting - or circumventing - these difficulties.

The foundation is then applied to develop a statistical theory of heterogeneous linearly elastic solids. The applicability of the theory for predicting the response characteristics of a class of ceramics, a type of composite, a polycrystalline solid, etc., is discussed. The practical utility of the theory is demonstrated.

PREFACE

This report was prepared by Professor John McCoy of Catholic University, Washington, D.C. while serving as a consultant at the Army Materials and Mechanics Research Center, Watertown, Massachusetts.

The work was completed for ARPA Order Number 2181, Program Element Code: 61101D, under the supervision of Dr. E. M. Lenoe, Mechanics Research Laboratory, AMMRC.

CONTENTS

1. INTRODUCTION

For an important class of engineering materials the homogeneous linearly elastic continuum represents an idealization that is valid only on a certain scale of observation, which may be termed the macroscale. A closer examination on a finer scale, which is still far above the atomic scale, reveals heterogeneity although the model of a linearly elastic continuum is still valid. This finer scale may be termed the microscale. Examples of such materials are numerous. The most important is possibly the polycrystal, which is an aggregate of a very large number of anisotropic crystals that are oriented in space in a random fashion. Each crystal is large enough to be idealized as a homogeneous linearly elastic continuum. Its mechanical properties are described by an elastic moduli tensor with components, referred to a space fixed system, that do not vary with position in the crystal. The components of the elastic moduli tensor for the polycrystal as a unit, again referred to a space fixed system, do vary with position in the polycrystal as one moves across the individual crystals. A second example of the class of materials of interest is the fiber reinforced composite. Here, again, the fibers or the regions of the matrix between fibers are large enough to be idealized by homogeneous linearly elastic continua. The parameters that define the mechanical properties do not vary with position in the fiber or in the matrix, but do vary with a position change from a fiber to the matrix.

The second feature that is common to the two examples cited is that the scalar fields needed to define the spatially varying material properties can only be described in statistical terms. That is, the scalar fields are given by stochastic processes. In this report we present the foundation of a theory that is capable of making predictions of such solids. While the development of a consistent statistical theory of randomly heterogeneous linearly elastic solids is relatively recent several interesting results have been achieved. To illustrate some of the more significant of these we present the following list.

1. The intuitively satisfying concept of the validity of using an effective modulus theory to predict the locally averaged response has been demonstrated for a restricted class of problems. The restrictions are that the locally averaged response fields vary slowly enough on a length scale defined by the substructure and that one does not make predictions in the immediate vicinities of forces or boundary surfaces.

2. It has been clearly demonstrated that the effective material properties depend on detailed information of the geometry of the substructure; i.e., shape information, clustering information, etc.; and not just on volume fraction information. Further, the statistical formulation provides a proper hierarchy for collecting this detailed information via the statistical moments.

3. It has been demonstrated that some corrections that should be applied to the effective modulus formulation to allow for a finite size substructure

can be obtained from field equations that can be formally identified with those that arise using strain gradient theories. No physical significance can be attached to the identification, however, since to achieve it one must violate thermodynamical requirements of the theories. Further, the strain gradient theories do not predict valid solutions for the immediate vicinities of forces and boundary surfaces.

4. A general formulation that incorporates a finite sized substructure and can validly make predictions for all regions of the specimen has been presented.

5. The validity of a dynamical effective modulus theory has been demonstrated. It has been shown that the dynamical effective elasticity tensor is the same as the statical effective elasticity modulus tensor. Also, it has been shown that the effective mass density can be properly identified with the averaged mass density.

6. The dynamical effective modulus theory predicts the nondispersive, nondecaying propagation of a signal. The statistical formulation demonstrates that both effects must be expected over long enough propagation distances. The apparent loss of energy implied by this statement is a result of incoherent scattering and not of the irreversible transformation of energy into a nonmechanical form.

7. A formulation has been presented in the form of a low frequency-long wavelength theory that incorporates both signal distortion and signal decay.

8. The possibility of the existence of high frequency-long wavelength solutions that are predicted by the microstructural theories has been shown to be remote.

9. A general formulation that would enable one to calculate terms such as $\langle(\tau-\langle\tau\rangle)^2\rangle$ where τ denotes the stress and $\langle\rangle$ denotes an averaging has been presented. This gives a measure of the average difference between the stress field and its locally averaged value. This is thought to be important in considering failure. No solutions of this formulation have yet been achieved.

Since a number of readers to which this report is directed may not possess the requisite familiarity with stochastic processes, we begin with some introductory sections. One section is devoted to the language of stochastic processes. Some care is taken in discussing the nature and the amount of information required to describe a random function. Emphasis is placed on the fact that although the information needed to differentiate between (or to equate) two random functions is different in kind from that needed to differentiate between (or to equate) two ordinary functions, it is not necessarily different in amount. This simple observation is crucial to an understanding of any problem involving a random input. Other sections are devoted to a discussion of a simple harmonic oscillator. In

one example we seek to show that the nature of the problem involved in predicting the response changes as the specific input parameter about which we have probabilistic information changes. We reserve the term statistical continuum theory to denote a class of problems for which the oscillator with a spring constant that depends on time in a random fashion is illustrative. The elasticity problem fits in this class. The nature of the statistical continuum problem is discussed in detail within the context of the illustrative spring-mass. Also, the results that are subsequently to be presented for the elasticity problem are derived within the context of the simpler illustration. In this way, it is hoped that the "physics" of the problem is not hidden in a mass of mathematical details.

After the introductory discussions, we turn to the physical problem of interest and present the outline of a statistical theory of heterogeneous linearly elastic solids. Due to an attempt to keep this section self-contained there is some repetition of ideas discussed in the introductory section. In a third section, we discuss the role of experimentation and numerical techniques in statistical continuum studies. In a final section we outline some ideas for areas of future studies that should prove to be fruitful.

2. PRELIMINARY DISCUSSION

2.1 ILLUSTRATIVE PROBLEM - I

Determining the response of a simple harmonic oscillator to a time dependent forcing can be used to illustrate several different types of problems involving a random input. The oscillator is a simple spring mass system and the mathematical problem that allows the determination of the response, i.e., $y(t)$, is given by the differential equation

$$\frac{d^2y}{dt^2} + k^2y = F(t) \tag{1}$$

together with the initial conditions

$$y(t_o) = y_o \tag{2a}$$

and

$$y(t_o) = v_o \tag{2b}$$

The input information required for a prediction of the response is contained in the variables k^2, $F(t)$, y_o, v_o and t_o. In the first class of problems the randomness, or uncertainty, enters by way of the forcing variable $F(t)$ or the initial values y_o and v_o. In this class, k^2 and t_o are taken to be known with certainty. In the second and third classes of problems the randomness enters by way of the spring constant k^2. The forcing and initial conditions are also known with certainty. The distinguishing feature separating the second

class from the third class is that in the second k^2 is a single number about which we have probabilistic information, whereas in the third $k^2(t)$ is a function about which we have probabilistic information. Finally, the fourth class of problems is characterized by the fact that the randomness enters by way of the time at which the initial conditions are specified. In this class k^2, $F(t)$, y_o and v_o are all taken to be known with certainty.

To motivate this system of classification, we first consider problems in which k^2 is a constant. This eliminates Class III from our discussion. Ignoring for the moment the nature of the information we have of any of our input variables, if k^2 is a constant we can write down a general solution that explicitly gives the dependence of $y(t)$ on k^2, $F(t)$, y_o, v_o and t_o. The reason for this is that, with k^2 equal to a constant, the mathematical problem requires the solution of a linear ordinary differential equation with constant coefficients. A general solution algorithm is known for such problems. We write the general expression

$$y(t) = \frac{v_o}{k} \sin k(t-t_o) + y_o \cos k(t-t_o) + \frac{1}{k} \int_{t_o}^{t} F(t_1) \sin k(t-t_1) dt_1 \qquad (3)$$

The greatest portion of the literature dealing with the response of an engineering system to a random input considers systems for which we are first able to construct a general expression explicitly relating the response variable to the input variables without taking into consideration the nature of the information of the input variables. It is only after a general expression like that given by Eq. (3) has been obtained that one introduces the fact that one of the inputs is known only in probabilistic terms. It is only in the more recent literature that one encounters papers that treat systems for which we are unable to proceed by first constructing the general inverse.

The general expression given by Eq. (3) emphasizes a fact that is also important in considering types of problems involving random inputs. This fact is that the dependence of the response, $y(t)$, on $F(t)$, v_o and y_o is linear whereas the dependence of $y(t)$ on k and t_o is nonlinear. That the response depends linearly on the forcing is, of course, to be expected since the original mathematical problem is obviously of the type that is classified as linear. It is sometimes overlooked, however, that the usual definition of linearity refers to only the relationship between the response variable and the forcing variables. The dependence of the response on "parameters" of the problem such as a coefficient in a differential equation or the instant that defines the initiation of the problem on the location of a boundary is usually nonlinear even for problems that are classified as linear. The consequences of the linearity or lack of linearity in

the dependence of the response variable on that particular input about which we have probabilistic information are great enough to warrant our differentiating between statistically linear problems and statistically nonlinear problems.

Let us now allow k^2 to vary with time. Again we ignore for a moment the nature of the information of our inputs. Since the problem is still of the type that is usually classified as linear, we may still write the following general expression

$$y(t) = y_o\, A(t) + v_o\, B(t) + \int_{t_o}^{t} G(t,t_1)\, F(t_o)\, dt_1 \tag{4}$$

where $A(t)$ and $B(t)$ denote the response of the oscillator to unit steps, applied at $t = t_o$, in displacement and velocity, respectively; and $G(t,t_1)$ is the response to an impulse load applied at $t = t_1$. Each of these terms, i.e., $A(t)$, $B(t)$ and $G(t,t_1)$, depend on the entire time history of $k^2(t)$ and the initial time t_o. They are independent of $F(t)$, y_o and v_o and the initial time t_o. Thus, for problems that fall into Class I, $A(t)$, $B(t)$, and $G(t,t_1)$ are deterministic terms. The fact that we can only write analytic expressions for them for simple variations of k^2 with time is less important than the fact that for a given $k^2(t)$ they can be obtained before we worry about the random nature of $F(t)$, y_o and v_o. For more complicated $k^2(t)$ one can resort to the digital computer. For problems that fall into Classes III and IV, on the other hand, $A(t)$, $B(t)$ and $G(t,t_1)$ are all random quantities. Further, we cannot write explicit expressions for the manner in which they depend on $k^2(t)$ and t_o. Thus, while the formal equation given by Eq. (4) may still be written, it is of no value in treating problems in Classes III and IV.

In summary, therefore, the simple oscillator illustrates several distinct problems in which the input data is random.

1. Random Forcing Problems (Linear) - The problem is to determine the response of a linear system to a random loading. The dependence of the response variable on the random input is linear and may be completely determined before any consideration is taken of the input. Random vibrations problems fall into this category. (1-2)

2. Random Forcing Problems (Nonlinear) - A natural generalization of the above. The dependence of the response variable on the random input is nonlinear and cannot be explicitly given in any application of interest. The theory of turbulence might be classified under this problem. The literature devoted to turbulence studies is vast and is a source of many of the ideas that have been tried in other areas. (3-5)

3. Randomized Parameter Problems - Problems in which it is possible

to explicitly relate the response variable to some system parameter that is described by a number about which we possess only probabilistic information. Class II problems fall into this category. Much of the work or reliability theory is of this type. (6)

4. Statistical Continuum Theories (Linear) - Mathematically, the problems are characterized by linear differential equations with coefficients that are random functions of the independent variables. Class III problems fall into this category. The dependence of the response variable on the random input is nonlinear and cannot be explicitly written. The theory of disordered composites, propagation through a random medium, dispersion of passive additives by a turbulent fluid, etc., all are problems of this type. The main portion of this paper is devoted to problems in this category.

5. Statistical Continuum Theories (Nonlinear) - A natural extension of the above that has received little, if any, attention.

6. Randomly Located Boundaries - A natural extension of Class IV problems. Flow through porous media and scattering by rough surfaces are two problems that may be categorized under this title. A few attempts have been made on predicting the response of solids in the vicinity of a rough boundary. (7-8)

We shall return to our illustrative problem after we first introduce some more precise terminology to replace the qualitative ideas of probability that have been used to date.

2.2 LANGUAGE OF STATISTICAL CONTINUUM THEORIES

The language of statistical continuum theories is rooted in the theories of probability and statistics and any attempt at serious work in the former area should be preceded by a good grounding in the latter. On the other hand, if one is willing to accept some lack of precision in considering details then one can go surprisingly far in understanding the physics of the problems using only intuitive ideas of probability. In this short section we attempt to rely on these intuitive ideas to motivate a language to use in discussing some physical problems.

Inherent to the theory of probability is an ensemble. A random spring constant is not a single number associated with a single spring but is a collection of a large number (i.e., infinite) of numbers associated with an ensemble, or collection of springs. A random spring constant is said to be known provided one can give the probability of choosing a spring, at random, from the collection and finding that the value of its constant lies between any and every pair of numbers that can be specified. Mathematically, this can be done by introducing a probability density function, $P_k(\xi)$, defined by the condition that the

probability of randomly choosing a spring that will have a constant with a value that falls somewhere between the values k=a and k=b is

$$\int_a^b P_k(\xi)\, d\xi^*$$

An alternate description of the random spring constant is given by its statistical moments. The nth order statistical moment of k, which we denote by $\langle k^n \rangle$ is defined by

$$\langle k^n \rangle = \xi^n p_k(\xi)\, d\xi \tag{5}$$

The lowest order moment, or $\langle k \rangle$, is termed the mean spring constant. Its physical significance is obvious. It is the weighted average of all spring constants defined by the ensemble. The second moment, or $\langle k^2 \rangle$ can be used to give a measure of the spread of values of all the spring constants about the mean value. We can write

$$\langle k'^2 \rangle = \langle k^2 \rangle - \langle k \rangle^2 \tag{6}$$

where $k' = k - \langle k \rangle$. The quantity $\langle k'^2 \rangle$ is termed the variance of the collection of spring constants. The higher order moments contain more refined information of the spread of the ensemble of values about the mean value.

One could introduce other alternate methods for describing the random spring constant, e.g., through a characteristic function that is defined by

$$P_k(S) = \int_{-\infty}^{\infty} P_k(\xi)\, e^{is\xi} d\xi \tag{7}$$

but for our purposes these latter will be of little value. One can obviously think up situations in which one manner of describing a random variable is superior to all others. For predicting reliability, the probability density function appears to be well suited. In the work to be described in this report the statistical moments have some very important advantages. Chief among these is that rarely, if ever, shall we be able to determine complete statistical knowledge of our random quantities. For many problems in which this is the case the statistical moments appear to collect partial information in a proper hierarchy of importance. Intuitively, the value of the mean spring constant is the most important information, a measure of the spread of the distribution of values about the mean is next, etc.

* Consistent with the purpose of the section some definitions and conclusions will lack the precision required by the rigorous formalism.

Turning from a proper description of a single constant with a value that is known only in probabilistic terms* we next consider a proper description of a pair of constants with values that are known only in probabilistic terms. As an example one might think of the Young's modulus, E, and the shear modulus, G, for a collection of homogeneous, isotropic, linearly elastic specimens. Taking the two quantities separately, each requires a description as discussed above. For example, one might give the two probability density functions $P_E(\xi)$ and $P_G(\xi)$. These two functions, however, do not constitute a complete statistical description of the pair of constants, since they fail to incorporate any type of constraint that the pair of values might be forced to satisfy. An example of a constraint for our ensemble of elasticity specimens arises if one is able, from other considerations, to place a value limitation on the Poisson's ratio, ν, of the specimens. This, and the known relationship that $G = E/2\ (1+\nu)$, places a limit on a pair of values to assign E and G. A complete statistical description of two random variables is given by a two dimensional joint probability function, say $P_{EG}(\xi,\eta)$. In the context of the example, $P_{EG}(\xi,\eta)$ is defined by the statement that the probability of randomly choosing a specimen that has both a Young's modulus with a value that falls somewhere between E=a and $\overline{E=b}$ and a shear modulus with a value that falls somewhere between G=c and $\overline{G}$=d is

$$\int_c^d \int_a^b P_{EG}\ (\xi,\eta)\ d\ \xi d\eta$$

The following statements may be proven from the definitions of

$$P_E\ (\xi),\ P_G(\eta) \text{ and } P_{EG}(\xi,\eta) \tag{8}$$

$$P_E\ (\xi) = \int_{-\infty}^{\infty} P_{EG}\ (\xi,\eta)\ d\eta,\ P_G(\eta) = \int_{-\infty}^{\infty} P_{EG}\ (\xi,\eta)\ d\xi$$

and, if, and only if, no constraints exist, i.e., E and G are independent, then

$$P_{EG}(\xi,\eta) = P_E\ (\xi)\ P_G(\eta) \tag{9}$$

We can conclude the discussion of a description of two random variables by noting that alternate, equivalent descriptions are possible.

Consider now a proper description of a function, say k(t), that is known only in probabilistic terms. Taking every instant of time singly, k(t) defines a random variable for each instant. Each random variable requires a description as discussed above. The totality of these descriptions is insufficient to completely categorize the function, however, for the reason discussed when considering two random variables. Taking every

* Should such a quantity be termed a random constant or a random variable? Consistent with the usual terminology we use random variable.

possible pair of instants together, k(t) defines a pair of random variables for each pair of instants. Each pair requires a joint probability density function as that introduced above. In addition, we must take every triplet (quartet, etc.) of instants together and consider the triplet (quartet, etc.) of random variables for each triplet (quartet, etc.) of instants. In this manner we obtain three (four, etc.) dimensional probability density functions. A complete statistical description of k(t) is the totality of all the information in all of the multi-dimensional probability density functions. It can be shown that the totality of this information can be given in a mathematical quantity termed a functional; in the present case, the probability density functional. One may think of a functional as a higher order function. A function is a rule for assigning one number, i.e., the value of the function, to another number, i.e., the value of the argument of the function. A functional is a rule for assigning one number, i.e., the value of the functional, to a continuum of numbers defined by a function, which is the argument of the functional. If we can accept the fact that an algebra and calculus of functionals have been developed, then we can imagine that the probability density functional gives the probability of finding a function within a range of functions by an integration in complete analogy to the probability density function giving the probability of finding a constant within a range of values by an integration. A mathematically proper formulation of problems involving the transformation of random processes would be in terms of functionals. Attempts at obtaining such formulation and then at looking for solutions using these formulations are very recent and have met with only limited success. The functional approach is well beyond the scope of the present report and will not be pursued further. The approach has only been introduced since it is important to keep sight of the nature of a properly formulated general problem.

We shall find an alternate description of a random function, via its statistical moments, to be useful. The one point statistical moments are given by

$$\langle k^n(t_1)\rangle = \int_{-\infty}^{\infty} \xi^n \, p_{k(t_1)}(\xi)\, d\xi \tag{10}$$

the two point statistical moments are given by

$$\langle k^n(t_1)\, k^m(t_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi^n \eta^m \, p_{k(t_1)k(t_2)}(\xi,\eta)\, d\xi d\eta \tag{11}$$

the three point statistical moments are given by

$$\langle k^n(t_1)k^m(t_2)k^p(t_3)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi^n \eta^m \zeta^p \, p_{k(t_1)k(t_2)k(t_3)}(\xi,\eta,\zeta)\, d\xi d\eta d\zeta \tag{12}$$

etc. A complete description of the stochastic process requires specification of all of the statistical moments. Clearly this is an enormous

amount of information that we shall be able to obtain only for some special situations, e.g., $k(t)$ is a Gaussian process in which case the higher order moments are uniquely determined by the one and two point moments. An advantage of the statistical moments is the observation that for stochastic processes arising in a variety of physical problems they appear to collect the information for a partial description of the process in the proper hierarchy of importance. Until very recently, the only higher order moment that has received much attention is the two point moment, $\langle k(t_1)k(t_2)\rangle$. The importance of the information contained in this two point moment is more apparent than that in the more complicated moments. Further, it will be seen that it is a more easily measured quantity than are the higher order moments. The most recent work indicates that the neglect of the higher order moments, and in particular of $\langle k(t_1)k(t_2)k(t_3)\rangle$ and $\langle k(t_1)k(t_2)k(t_3)k(t_4)\rangle$ is to end. This point shall be emphasized further during the discussion of some physical problems.

One should not, however, jump to the conclusion that collecting information via the statistical moments is the most efficient or convenient for all physical problems. The most convenient collection depends on the physical problem to be studied. Researchers on the fatigue of specimens subjected to random loading appear to have found it convenient to categorize loadings according to probability distributions defined on the extremum values and on the zero crossings. For other physical problems one can undoubtedly devise still other description procedures. One should always keep in mind, however, that a complete statistical description requires all of the information contained in the probability density functional, or, equivalently all of the information in all of the multi-dimensional probability density functions. Anything less than all of this information implies that there is a nonuniqueness in our description of the random process. In general we shall see that a nonuniqueness in the description of a random input to a problem must lead to a nonuniqueness in the description of the random output. Realization of this nonuniqueness is crucial in the context of some physical problems and shall be again discussed in a later section.

At this point it is convenient to introduce the concept of a stationary random process, or function of time. A process is termed stationary if all of the multi-dimensional probability density functions needed to define the process are independent of absolute time*. Thus $P_{k(t)}(\xi)$ for example, is independent of time; $P_{k(t_1)k(t_2)}(\xi,\eta)$ depends only on the time difference

$$t_1-t_2,\quad P_{k(t_1)k(t_2)k(t_3)}^{(\xi,\eta,\zeta)}$$

depends only on the time differences t_1-t_2 and t_1-t_3, etc. In terms of the statistical moments, stationarity implies that $\langle k\rangle$ will be a constant that $\langle k(t_1)k(t_2)\rangle = \sigma^2(\tau)$, where $\tau = t_1-t_2$, that $\langle k(t_1)k(t_2)k(t_3)\rangle = N_3(t_1-t_2,t_1-t_3)$ etc. Strictly speaking stationarity can only be achieved by processes of unending duration. For many real processes, however, it is possible to

*It is common to differentiate between a wide sense stationary process as given in this paper and a less restricted definition that only figures the one and two point probabilities to be independent of absolute time.

view stationarity as an ideal that becomes a better approximation the longer the duration of the process relative to all other characteristic times of the process. As an example of a characteristic time, and the one to which reference is most frequently made, we consider the probability density defined on $k(t)$ measured at a pair of times, say t_1 and t_2. For the majority of stochastic processes that arise in physical problems any constraints that must be satisfied by a pair of values $k(t_1)$ and $k(t_2)$ will exist only for time differences, t_1-t_2, that are less than some maximum value. For times separated by a greater interval,

$$P_{k(t_1)k(t_2)}^{(\xi,\eta)} = P_{k(t_1)}(\xi)\, P_{k(t_2)}(N).$$

This maximum time interval is a characteristic of the process that must be much shorter than the duration of the process for the concept of stationarity to have much meaning. In the literature it is common to assume that all characteristic times of this type are of the same order of magnitude and to use the time interval required in order to insure that $\langle k(t_1)k(t_2)\rangle = \langle k(t_1)\rangle\langle k(t_2)\rangle$ as a measure of this order of magnitude.

For stationary processes we can now introduce the concept of ergodicity, which is of fundamental importance for some of the interpretations of the work to be described in this paper. Basically, an ergodic hypothesis is a statement that if the statistics of the process are independent of absolute time then to obtain a given statistical measure we are justified in using different values of absolute time in the same record in place of the same value of absolute time in different records. Thus, for example, in order to obtain the probability density function for $k(t)$ measured at a single instant of time we can, for an ergodic process, replace the sampling that is required at that same instant of time on many records by a sampling that is to be taken at many instants of time in the same record. The importance of an ergodic process is that we can obtain a complete statistical description of the process from a single manifestation of the process. As an intuitive concept to be applied to some physical systems the validity of ergodicity is often apparent. A mathematically rigorous justification of it is rarely possible. In the work to be presented, the concept of ergodicity does not enter any of the results achieved. It is, however, important in interpreting some of these results. This aspect will be further discussed within the context of specific physical problems.

In terms of the statistical averages, ergodicity implies the equality of a statistical average and a temporal average. Thus, for example,

$$\langle k\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} k(t)\,dt = \text{constant} \qquad (13)$$

and

$$\langle k(t)k(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} k(t)k(t+\tau)\,dt = \sigma(\tau) \qquad (14)$$

etc. The time average of the product of a record with a shifted replica of itself has been a quantity used by electrical engineers for many years. In their terminology it has been called the correlation function. We might note that it is in interpreting a statistical average by a temporal average or by a spatial average that we shall be invoking an ergodic hypothesis.

One final concept that must be introduced to obtain even a cursory understanding of some of the work involving statistical continuum theories is that of an inverse space or Fourier representation. The power of Fourier analysis in deterministic situations necessitates that all attempts be made to extend it to stochastic functions. An extension is really not necessary for processes of finite duration. For stationary processes, however, an extension is required and it is this extension that is the subject matter of the Wiener-Khinchin theorem. Without going into any proofs, which are quite complicated, the Wiener-Khinchin theorem tells us that

1. A Fourier representation of all stationary random processes exist although it is necessary to define it in terms of Stieljes integrals.

2. The amplitude spectrum defined by the Fourier representation of the stationary random process is deterministic. All of the indeterminancy of the process is contained in the phase spectrum.

3. Locally averaged (in frequency space) amplitude spectrum squared and the two point moment or correlation function are related to each other as Fourier transform pairs.

To obtain some intuitive insight into the meaning of the Wiener-Khinchin theorem, consider the stationary random process to be an electrical signal. In this context one can give a procedure for directly measuring the locally averaged amplitude spectrum squared. The value of the locally averaged amplitude spectrum squared at a given frequency is equal to the power of the signal obtained by passing our original electrical signal through a narrow band filter centered at this frequency. The locally averaged amplitude spectrum squared is, thus, termed the power spectrum in the literature. The Wiener-Khinchin theorem informs us that the power spectrum and the correlation function are Fourier transform pairs. It was the ability of the electrical engineers to produce relatively simple and inexpensive circuits to directly measure power spectra that resulted in the interest in these spectra and in their transform pairs, the correlation functions. More refined measures of stationary processes have been neglected largely because of the absence of measurement techniques. As a result of the increase in digital processing of data, for which we can theoretically obtain any measures we desire, interest is now being shown in the more refined measures.

While all of the preceding discussion of stochastic processes illustrated the process as a function of time, it is not very difficult

to translate it to processes that are functions of position. The work to be discussed in this report will deal with functions of position. For one dimensional problems the translation only requires a change in terminology. Stationarity is now termed statistical homogeneity. The concept of an inverse space, or k space, is not as familiar to the mechanics researcher who might be interested in the one dimensional function of position as is the frequency space to the electrical engineer. This is really somewhat unfortunate since it would be equally as powerful in the mechanics problem and can be used to the same purpose. The spectrum of a mechanical signal really amounts to a resolution of the signal into components which make additive contributions to the energy in the signal. Further, one can attach an easily recognized physical meaning to the components since they correspond to contributions of different linear size. For three dimensional problems some extension is necessary. In addition to statistical homogeneity we can introduce some invariance properties of the statistics under some rotations. Statistical isotropy requires invariance of the statistics for all rotations. For ergodic processes, statistical averages will be equated to surface or volume averages. Some extension is required in the theory behind the Fourier decomposition of the process. This has been accomplished. It is even possible to extend the one dimensional analog filter of the electrical engineer into two dimensions by using optical techniques. The versatility of digital processing of data, however, would indicate that it will probably receive increasing attention in future studies.

With this background into the language of stochastic processes we are ready to reconsider our illustrative example.

2.3 ILLUSTRATIVE PROBLEM II

Now that we have a precise definition of the information required to describe a constant or a function in statistical terms we can return to the illustrative problem to gain some further insight into the nature of physical problems that deal with input data that can only be described in such terms.

Before we make any attempt at solving the problem posed we should first assure ourselves that the problem posed does possess a solution. We should, if possible, also like to assure ourselves that the problem possesses a single solution. Should this latter be impossible we should like to at least be able to quantify any nonuniqueness that may exist. There should be no need to make excuses for raising questions as to existence and uniqueness of solution and none will be given here. Neither shall we indulge in long formal proofs since it is beyond both the level at which we discussed the description of a stochastic process and the level of this report. We should, however, like to make what can be shown to be a properly posed problem appear to be reasonable and further show that many problems that are implicitly stated do not possess a unique solution.

In simplified terms the point that is to be emphasized may be presented as follows: Consider the deterministic spring-mass system. From our physical intuition, or past experience, or a formal mathematical proof, we know that the response history is uniquely determined by the initial conditions and the histories of the spring constant and the forcing. Suppose, however, that we do not know the complete history of the spring constant. Suppose that gaps exist during which time we have no knowledge of k(t). What can now be said of the response history? Obviously, we cannot still determine it or the original problem would have been over specified. Can we assume that the response history is still determined except within the time gaps during which we have no information of k(t)? Our knowledge of the spring-mass system is sufficient for us to conclude that this too is incorrect. The random input problem is analogous to this. We no longer have a complete deterministic history of k(t), for example, but have a probability density functional defined on k(t). It is intuitively clear that this information is insufficient for us to determine a unique deterministic history for y(t). Is it sufficient to uniquely determine a probability density functional defined on y(t)? We suspect that the answer is yes. We could imagine that we could take each of the spring histories in the ensemble defined by the probability density functional and separately determine the response for each ignoring the statistical nature of the original problem. This gives us the desired ensemble of response histories and it is not very difficult to see that the probability of choosing one may also be obtained. It is possible to rigorously justify the suspicion that complete information of the probability density functional defined as k(t) is sufficient to uniquely determine complete information of the probability density functional defined on y(t). What about partial information? For instance, suppose we only know $\langle k(t) \rangle$. This is obviously not enough information to uniquely determine a complete statistical description of y(t) but is it enough to uniquely define $\langle y(t) \rangle$? The answer to this question is no! This answer is a direct consequence of the nonlinearity in the dependence of y(t) on k(t). We shall later show that any attempt to uniquely relate an effective material property of a two phase composite to the volume fraction of the constiuent phases is predicated on a erroneous assumption that is identical to assuming that $\langle y(t) \rangle$ is uniquely determined by $\langle k(t) \rangle$. Numerous examples of this same erroneous assumption taken from a variety of areas of study could likewise be listed.

We can use the general solution for the oscillator with a spring constant that is independent of time to illustrate the assertion as to the relationship between lack of uniqueness and lack of linearity. Let us take $v_o = y_o = 0$. First we consider the problem in which F(t) is the random input. We can obtain an expression for $\langle y(t) \rangle$ by averaging Eq. (3). Since the averaging and integrating operations are both linear the orders of performance may be interchanged. The result is

$$\langle y(t) \rangle = \frac{1}{k} \int_{t_o}^{t} \langle F(t_1) \rangle \sin k (t-t_1)\, dt_1 \qquad (15)$$

Thus, $\langle y(t)\rangle$ is uniquely defined by $\langle F(t_1)\rangle$. Consider next the problem in which k is the random input. Again we average Eq. (3) to obtain

$$\langle y(t)\rangle = \int_{t_o}^{t} F(t_1)\left\langle \frac{1}{k} \sin k\,(t-t_1)\right\rangle dt_1 \tag{16}$$

The nonlinearity in the operations of taking an inverse, taking a sine, or taking a product prevents our interchanging their order of performance with that of taking an average. Equation (16) clearly demonstrates that knowledge of $\langle k\rangle$ is insufficient to determine $\langle y(t)\rangle$. Indeed, expanding the sine function as a power series enables us to see that if the statistical information of k is described by its statistical moments then knowledge of all of them is required to predict $\langle y(t)\rangle$.

It is instructive to consider $\langle y^2(t)\rangle$ from which we can calculate the average of the square of the differences between y(t) and $\langle y(t)\rangle$. Squaring Eq. (3) with $v_o = y_o = 0$ and averaging the result leads to - in the random forcing problem -

$$\langle y^2(t)\rangle = \frac{1}{k^2} \int_{t_o}^{t}\int_{t_o}^{t} \langle F(t_1)\,F(t_2)\rangle \sin k\,(t-t_1) \sin k\,(t-t_2)\, dt_1 dt_2 \tag{17}$$

We see from Eq. (17) that the full two point moment of the forcing is required to determine the variance of the response.

For k dependent on time we look to the general expression given by Eq. (4), with $X_o = Y_o = 0$

$$y(t) = \int_{t_o}^{t} G(t,t_1)\,F(t_1)\,dt_1 \tag{18}$$

Once again, we note that a general analytic expression that gives the dependence of the impulse response function, i.e., $G(t,t_1)$ on the time history for k(t) is not known. For a uniquely specified k(t), however, we can reasonably expect that $G(t,t_1)$ can be constructed by means of a numerical algorithm. Or, since we are dealing with a uniquely defined physical system we can model it in the laboratory and experimentally determine $G(t,t_1)$. With $G(t,t_1)$ known, Eq. (18) gives the response for any loading history. For random loading histories, we can use Eq. (18) to obtain the following deterministic equations on the most important averages defined on the response,

$$\langle y(t)\rangle = \int_{t_o}^{t} G(t,t')\ \langle F(t')\rangle\, dt' \tag{19}$$

and

$$\langle y(t_1)y(t_2)\rangle = \int_{t_o}^{t}\int_{t_o}^{t} G(t_1,t')\,G(t_1,t'')\langle F(t')F(t'')\rangle dt'\,dt''$$

For k(t) a stochastic process one can still imagine a numerical or experimental determination of $G(t,t_1)$. Now, however, the program requires

that we deal with ensembles and not a single system. In this way, we can determine all desired information of $G(t,t_1)$. (We shall discuss the numerical problem in a later section.) Equation (18) can again be used to determine averages defined on the response for any loading history. For uniquely specified histories we may write

$$\langle y(t)\rangle = \int_{t_o}^{t} \langle G(t,t')\rangle F(t')\, dt' \tag{20a}$$

and

$$\langle y(t_1)\, y(t_2)\rangle = \int_{t_o}^{t}\int_{t_o}^{t} \langle G(t_1,t')G(t_2,t')\rangle\, F(t')F(t'')dt'dt'' \tag{20b}$$

where $\langle G(t,t')\rangle$ and $\langle G(t_1,t')G(t_2,t')\rangle$ can be determined by the Monte Carlo type approach discussed.

We can speculate on another possible fruitful way to use equations of the type given by Eq. (20) that would enable our by-passing the Monte-Carlo procedure. Consider Eq. (20a), for example, and imagine that it is possible to define a well posed inverse for it. That is, if the direct problem given by Eq. (20a) is to determine $\langle y(t)\rangle$ from information of $\langle G(t,t')\rangle$ and $F(t')$, then the inverse problem is to determine $\langle G(t,t')\rangle$ from information of $\langle y(t)\rangle$ and $F(t')$. Assuming the inverse problem exists, it is then possible to interpret the single deterministic equation, i.e., Eq. (20a), as a phenomenological model that is capable of predicting the mean field response under any loading history. Phenomenological in the sense that the model contains phenomenological parameters that are to be determined by requiring agreement between predictions of the model and experiments in some well defined test cases. The procedure for using the model is a two step procedure.

1. For the test situations, obtain the information of $\langle y(t)\rangle$ and $F(t)$ that is required by the inverse problem. Solve the inverse problem for $\langle G(t,t')\rangle$.

2. With $\langle G(t,t')\rangle$ known, the direct problem then gives the mean response for any loading history.

Parenthetically we note that the classical elasticity theory is a phenomenological model with the constants of elasticity as phenomenological parameters. All of the higher order elasticity theories, i.e., micropolar, micromorphic, strain gradient, etc. are likewise phenomenological models with greater numbers of phenomenological parameters.

It is via phenomenological models, although not those given by Eqs. (20a. and 20b), that the author believes we shall develop an ability to make predictions of $\langle y(t)\rangle$, $\langle y(t_1)\, y(t_2)\rangle$, or the higher moments for problems in which $k(t)$ is the stochastic input. In the next section

we develop models for $\langle y(t) \rangle$ and $\langle y(t_1)\, y(t_2) \rangle$ that hold promise of being particularly fruitful. In addition, the required extension for models on higher order moments shall be clear. Once the models have been developed we shall be able to anticipate the form of similarly developed models for the random linearly elastic solid.

One final remark may be made before leaving this section. For problems in which the random variations of $k(t)$ about a constant mean value, i.e., $\langle k \rangle$, are small relative to the mean value, it is possible to obtain analytic expressions for $G(t,t')$ as a functional of $k(t)$ by resorting to a perturbation analysis. For these same problems, our phenomenological models cease to be phenomenological in that we shall now have analytic expressions defining what would be the phenomenological parameters. It should be expected that the approximation obtained by a straight forward perturbation analysis could differ from that given by our models since the convergence properties of the two approaches could differ. This is indeed the case. For such cases arguments can be presented that indicate that the convergence properties of the approximation given by our model are better. We shall not consider the weakly random spring in detail since it is not representative of the important physical problems involving heterogeneous solids.

2.4 PHENOMENOLOGICAL MODELS GOVERNING MEAN FIELDS AND CORRELATION FUNCTIONS

We present here a formal mathematical development of equations that must be satisfied by $\langle y(t) \rangle$ and $\langle y(t_1)\, y(t_2) \rangle$. These equations then serve as the basis for our phenomenological models. No apology need be made for any real or imagined lack of rigor in the development procedure since the validity of the end result, like that of any phenomenological model, is determined by the ability to reproduce experimentally verifiable results. The author believes that a rigorous mathematical justification will one day be forthcoming for some restrictive situations.

We again write Eqs. (1) and (2)

$$\frac{d^2y}{dt^2} + k^2 y = F(t) \tag{1}$$

$$y(t_o) = y_o \tag{2a}$$

$$y(t_o) = v_o \tag{2b}$$

where $k^2(t)$ is taken to be described by a random function of position. We first consider the development of an equation to be satisfied $\langle y(t) \rangle$.

The procedure used is termed the method of smoothing (9-13). Averaging the basic equations gives

$$\frac{d^2\langle y\rangle}{dt^2} + \langle k^2\rangle \langle y\rangle + \langle (k^2)' y'\rangle = F(t) \tag{21}$$

$$\langle y(t_o)\rangle = y_o \tag{22a}$$

and

$$\langle y(t_o)\rangle = v_o \tag{22b}$$

A prime has been introduced to denote the difference between a random variable and the mean value, i.e.,

$$(k^2)' = k^2 - \langle k^2\rangle$$

and

$$y' = y - \langle y\rangle \tag{23}$$

By definition, $\langle (k^2)'\rangle \equiv \langle y'\rangle \equiv 0$. Statistical dependence of the response $y(t)$ and $k^2(t)$ prevents the conclusion that

$$\langle (k^2)' y'\rangle = \langle (k^2)'\rangle \langle y'\rangle = 0$$

Equations (21) and (22) do not, as they stand, serve as the desired equations on $\langle y(t)\rangle$ since, in addition to this quantity, they contain the unknown $\langle (k^2)' y'\rangle$. They will become the desired set of equations once we obtain a functional dependence of $\langle (k^2)' y'\rangle$ on $\langle y\rangle$. To this end, we subtract the averaged equations, i.e., Eqs. (21) and (22), from the basic equations to obtain

$$\frac{d^2y'}{dt^2} + \langle k^2\rangle y' + [(k^2)' y' - \langle (k^2)' y'\rangle] = -(k^2)' \langle y\rangle \tag{24}$$

and

$$y'(t_o) = 0 \tag{25a}$$

$$y'(t_o) = 0 \tag{25b}$$

Equations (24), and (25) are now viewed as conditions that must be satisfied by $y'(t)$. By direct substitution one can show that they will be satisfied by the infinite series

$$y'(t) = \sum_{n=t} y'^{(n)}(t) \tag{26}$$

Provided the series converges*, provided for $n = 1$,

$$\frac{d^2y'^{(1)}}{dt^2} + \langle k^2\rangle y'^{(1)} = -(k^2)' \langle y\rangle \tag{27}$$

*It is the convergence properties of the infinite series that must be determined for the development procedure to enjoy the status of being mathematically rigorous.

$$y'^{(1)}(t_o) = 0 \tag{28a}$$

and provided for

$$n > 1, \tag{28b}$$

$$\frac{d^2 y'^{(n)}}{dt^2} + \langle k^2 \rangle y'^{(n)} = \langle (k^2)' y'^{(n-1)} \rangle - (k^2)' y'^{(n-1)} \tag{29}$$

$$y'^{(n)}(t_o) = 0 \tag{30a}$$

$$y'^{(n)}(t_o) = 0 \tag{30b}$$

Equations (27) thru (30) are to be satisfied in sequence - first for $y'^{(1)}$, then $y'^{(2)}$, etc. The problem to be solved for each $y'^{(n)}$ is recognized to be that of determining the response of a spring-mass system, with spring constant equal to $\langle k \rangle$, to a known forcing. The initial conditions to be applied are zero. The solution of each is immediately given by Eq. (3) - for $\langle k^2 \rangle$ a constant - or by Eq. (4) - for a more general $\langle k^2 \rangle$. With all $y'^{(n)}(t)$ thus determined, we can form $\langle (k^2)'y' \rangle$ in terms of an infinite series by making use of Eq. (26). We substitute the result into Eq. (21) which leads to the desired phenomenological model on $\langle y(t) \rangle$. We write the result as:

$$\frac{d^2 \langle y \rangle}{dt^2} + K(t,t_1) \langle y(t_1) \rangle \, dt_1 = F(t) \tag{31}$$

and

$$\langle y(t_o) \rangle = y_o \tag{32a}$$

and

$$\langle y(t_o) \rangle = v_o \tag{32b}$$

The term $\langle k^2 \rangle \langle y \rangle$ has been incorporated into the kernel function by introducing the Dirac Function. The infinite series that was encountered in the development procedure is all contained in an infinite series definition for the kernel function. With an explicit algorithm for evaluating the kernal function there is nothing phenomenological about the model. It is only when we choose to ignore the explicit algorithm (which is useless until we are able to truncate or sum the infinite series) and elect to determine $K(t,t_1)$ by matching predictions of the equations with experimental data that the model becomes phenomenological. We have implicitly assumed, of course, that the inverse problem of determining $K(t,t_1)$ from measurements to be made of $\langle y(t) \rangle$ and $F(t)$ is well-posed. This question as to the measurability of $K(t,t_1)$ has not received a great deal of attention within the context of the general model and we shall not discuss the question here. We can note, however, that measurability has been

demonstrated in some applications (see below) and can be trivially demonstrated for some approximate phenomenological models that can be derived from Eq. (31).

The utility of Eq. (31) as a phenomenological model is severely limited by the large degree of arbitrariness for the kernel function, $K(t,t_1)$. For most physical problems of which our example is to be illustrative, the degree of arbitrariness for $K(t,t_1)$ is probably far too great for us to ever hope that we can completely determine it in the laboratory. To obtain phenomenological models that should prove to be of computational usefulness it will be necessary to place restrictions on $K(t,t_1)$. Some of the restrictions may result from general requirements as to causality, invariance groups and the like, some may result by the requirement that $K(t,t_1)$ not contradict the infinite series prescription, and some might arise as a result of speculation based on physical intuition.* The ultimate proof of these last mentioned restrictions will be the ability of the resulting restricted models to reproduce experimental results. Only recently has some effort been spent in studying the consequences of restricted forms for $K(t,t_1)$ and this effort has largely been devoted to speculative restrictions. We can consider some of the consequences of the most obvious attempts at speculation.

To motivate some speculations we consider the weakly random spring. The infinite series prescription for $K(t,t_1)$ is collected in terms of powers of the strength of the variations in $k^2(t)$. Thus, for a weakly random spring, it is valid to truncate the series to obtain a completely prescribed kernel. To first order we write

$$K(t,t_1) = \langle k \rangle \, \delta(t-t_1) + \frac{1}{\langle k \rangle} H(t_1-t_o) \sin\{\langle k \rangle (t-t_1)\} \sigma(t-t_1) \qquad (33)$$

where $\delta(t-t_1)$ is the Dirac Function, $H(t_1-t_o)$ is the Heaviside function and

$$\sigma(t-t_1) = \langle (k^2(t))' \, (k^2(t_1))' \rangle \qquad (34)$$

Implicit in Eqs. (33) and (34) is a limitation to stationary statistics, which is the case of greatest practical interest. The weakly random spring is illustrative of some important practical problems such as the scattering of a radiation field by a turbulent fluid. For problems involving the response of heterogeneous solids, however, its usefulness is limited to a less direct application. The direct use of Eq. (33) is

*It is possible to note an identity of form between the model derived and the equations that govern the response of a spring with memory. That is, a spring that exerts a restoring force that depends on the entire past history of its extension and not just on its current value. Once said, however, one should not attach too much significance to this formal identity. In particular, it would not be correct to introduce thermodynamic considerations that would be applicable to a real spring with memory to limit forms of the kernel function. It must be remembered that $\langle y(t) \rangle$ must be interpreted in terms of some type of average and that $y'(t)$ can offer a contribution to an average energy in the spring since energy depends on a power of the deformation.

prevented by the observation that most important heterogeneous solids are not weakly random. The indirect usefulness of Eq. (33) is predicated on the speculation that some conclusions drawn for the weakly random form for $K(t,t_1)$ may prove to have a wider range of validity. For example, we can note that to be illustrative of most problems of physical interest the stochastic process, $k^2(t)$, is statistically independent of a time shifted replica of itself if the time shift is greater than some maximum value, say τ. Thus, $\sigma(t-t_1) = 0$ for $t-t_1 > \tau$. Equation (33) indicates that non-zero values of $K(t,t_1)$ will likewise be confined to time instants that are within a range of τ of one another. One might speculate that this observation is likewise true for the strongly random spring. Also, we can note from Eq. (33) the $K(t,t_1)$ depends not only on the statistics of $k^2(t)$ but also on the domain of the original problem specification. Subject to the above described process, however, we can further note that any dependence of $K(t,t_1)$ on t_o is confined to pairs of time instants t,t_1 that fall within an interval of duration τ of t_o. If the observation time for the response is sufficiently long relative to τ we might reasonably obtain an approximation by ignoring this time interval. This allows us to view $K(t,t_1)$ as a property of the ensemble of springs alone, which is very desirable. It also allows us to take $K(t,t_1)$ to be a function of the time difference alone. Assuming the correctness of these speculations, a restricted form of $K(t,t_1)$ will result in a greatly simplified phenomenological model. Indeed, we can note that with $K(t,t_1)$ a function of the time difference alone the integral in Eq. (33) is of the convolution type, which suggests transforming the entire problem into frequency space. The frequency space representation of the solution is readily given

$$\langle \tilde{y}(\omega) \rangle = \frac{\tilde{F}(\omega) + v_o - i\omega y_o}{\tilde{K}(\omega) - \omega^2} \tag{35}$$

where a tilda denotes the Fourier representation. Equation (35) can be compared with the frequency space representation of the response of a time independent linear spring. The difference is that $\tilde{K}(\omega) = k^2$ in the latter case. Equation (35) represents the general solution of the restricted phenomenological model. Recalling the speculations leading to the restricted model we cannot expect that $\langle \tilde{y}(\omega) \rangle$ will accurately predict the mean response in the immediate vicinity of the boundary of the domain of the problem, i.e., near $\tau = t_o$. We might note that Eq. (35) proves the measurability of $K(t-t_1)$ for the restricted model.

Although a general solution has been achieved for the restricted model, it is instructive to return to the formulation and introduce some different approximations. In this way, we can investigate the validity of some other phenomenological models as they apply to predicting the average response of a random spring. The approximations to be introduced are thought to be valid for treating problems in which τ is very short relative to $\langle k^2 \rangle^{-1/2}$ and to all characteristic times defined by $F(t)$, y_o and v_o. We let T denote the shortest of these times and limit

consideration to problems for which $\tau/_T << 1$. It seems reasonable to assume that variations in $\langle y(t) \rangle$ are measurable on a time scale of order T. As a zeroth order approximation, therefore, we might neglect all variations in $\langle y(t) \rangle$ measured over a time interval of order τ. This allows removal of $\langle y(t) \rangle$ from under the integral sign of Eq. (31) giving

$$\frac{d^2 \langle y \rangle}{d\tau^2} + k^{*2} \, y = F(t) \tag{36}$$

where

$$k^{*2} = {}_o\int^{\infty} K(\xi) \, d\xi \tag{37}$$

Thus, for problems in which we can identify two widely differing time scales, one defined by the detailed variations in the stochastic process and one defined by the averaged characteristics of the system and by the forcing, the averaged response is approximated by the response of an "effective" spring-mass. We might further note that it is this identical condition, i.e., the presence of two length scales, that enables our taking recourse to ergodicity and interpreting $\langle y(t) \rangle$ as a temporal average. The time interval for the average is long relative to τ and short relative to T. Use of the classical elasticity theory to make predictions of the "averaged" response of a polycrystalline material, which is an example of a random solid, is based on assumptions as to the validity of both the ergodic hypothesis - in this case we equate an ensemble average with a volume average - and the effective modulus concept.

It is to be noted that the solutions predicted by the effective modulus for $\langle y(t) \rangle$ predict that variations in $\langle y(t) \rangle$ are measurable on the same time scale as variations in the forcing, i.e., on the T time scale. Thus, the effective modulus formulation is self-consistent in that the results achieved from the approximation are in agreement with the approximation. While self-consistency is an intuitively satisfying requirement of an approximate formulation it is by no means rigorous. Self consistency could merely mean that the approximate formulation only reproduces (approximately) a limited subset of the solutions of the exact formulation and that the members of that subset satisfy the conditions on which the approximation is based. To see this we can resort to the general solution given by Eq. (35). Resorting to residue theory, the real time solutions, $\langle y(t) \rangle$ arise due to pole type singularities for $\langle \tilde{y}(\omega) \rangle$. These are given by the forcing via, $\tilde{F}(\omega)$, and by $K(\omega) - \omega^2$. Real time solutions that vary slowly relative to $1/\tau$ correspond to singularities located in the region of frequency space that satisfies $\omega << 1/\tau$. In frequency space, the effective modulus formulation amounts to replacing $\tilde{K}(\omega)$ by its value at $\omega = o$, i.e., by k^{*2}. This

approximation results in a shift in the singularities that are located in the vicinity of $\omega = \pm k^*$. It also changes the values of the residues for $\langle \tilde{y}(\omega) \rangle$ for these poles and for the poles located by $\tilde{F}(\omega)$. These changes may all be viewed as modifications of corresponding solutions of the exact equations. In addition, this approximation serves to obliterate additional poles arising from $K(\omega) - \omega^2 = 0$. Thus, the exact formulation admits of solutions that have no counterpart in the effective modulus formulation. Whether these solutions will be physically significant will depend on the individual problem that is of interest.

One can attempt to obtain an improvement to the effective modulus formulation that allows a degree of memory by expanding $\langle y(t) \rangle$ as it appears in the integral in Eq. (31) in terms of a power series about $\langle y(t) \rangle$ and in truncating after the linear term. The first order approximation is thus given by

$$\frac{d^2 y}{dt^2} + k^{*2} \tau_1 \frac{d\langle y \rangle}{d\tau} + k^{*2} \langle y \rangle = F(t) \tag{38}$$

where

$$\tau_1 = \frac{1}{k^{*2}} \int_0^\infty \xi K(\xi)\, d\xi \tag{39}$$

It is to be expected that τ_1 is of the same order of magnitude as is τ. Equation (38) may be recognized as the equation that governs the response of a slightly damped spring-mass. What is the dissipation mechanism in a system that started out as being conservative? The answer lies in a statement presented in a previous footnote. The dissipation type term arises not due to an irreversible transformation of mechanical energy to nonmechanical energy but to an irreversible transformation of coherent mechanical energy to incoherent mechanical energy. That is the energy is taken from a response history that has a non-zero average and transformed into a response history that has a zero average.

Making reference to the general solution given by Eq. (35), we can conclude that the first order approximation amounts to replacing $K(\omega)$ by the first two terms of its power series expansion about the origin. Retaining the linear term in the expansion results in a shift in the singularities that the effective modulus approximation locates at $\omega = \pm k^*$. It also changes the values of the residues for all poles. It does not introduce additional poles, however. Presumably the values of all these quantities as predicted by the first order approximation will be closer to the exact values by an amount of order (τ/T). The real time effect of changing the location of a pole grows with the observation time. The first order correction results in a shift of the pole from $\omega = \pm k^*$ to a value that lies below the real frequency axis, for $\tau_1 > 0$. In real time, this shift corresponds to the introduction of an exponential decay. This is consistent with the observation made between the identity of Eq. (38) and the equation for a damped spring-mass. The real time

effect of changing the value of a residue of a simple pole does not grow with observation time.

In summary, then, the most significant conclusion in comparing the real time solutions of the zeroth and first order approximations is that the differences are not uniform in observation time. One must suspect, therefore, that both approximations will cease to be valid for long enough observation times. The expectation is for the first order solutions to be valid for a longer observation time.

Higher order formulations can be obtained by keeping more terms in the power series. In real time this results in higher order differential equations. In the frequency space solution it is mirrored by a higher order polynominal for $K(\omega) - \omega^2$. For approximations of order three or greater the effect of retaining still one more term is to modify solutions that have counterparts in the lower order formulations and to introduce additional solutions. All of these solutions that have no counterpart in the effective modulus solution are not consistent with the reasoning justifying the truncation. That is, they all vary on a scale of characteristic time equal to τ. Because of this, the solutions must be treated with suspicion. This does not mean, of course, that Eq. (31) can never admit of solutions that vary on a time scale of order τ (Eq. (35) indicates that such solutions will exist), it means rather that we are required to retain the complete integro-differential equation formulation if we are to accurately predict them. We might also emphasize that any solutions for $\langle y(t) \rangle$ that vary on a time scale of order τ can only be interpreted in terms of an ensemble average.

In a later section we shall consider the conclusions reached here for the illustrative problem as they apply to the theory of heterogeneous solids.

We next turn to the development of an equation to be satisfied by $\langle y(t)\, y(\xi) \rangle$. First we multiply Eq. (1) taken at the point τ by the same equation taken at the point ξ. The result is written

$$\left[\frac{d^2}{dt^2} + k^2(t)\right]\left[\frac{d^2}{d\xi^2} + k^2(\xi)\right] y(t) y(\xi) = F(t)F(\xi) \tag{41}$$

A slightly modified method of smoothing (14) is now used in conjunction with Eq. (41). The final equation is written as

$$\left(\frac{d^2}{dt^2} + \langle k^2 \rangle\right)\left(\frac{d^2}{d^2} + \langle k^2 \rangle\right) \langle y(t) y(\xi) \rangle$$

$$+ \left[\left(\frac{d^2}{d\tau^2} + \langle k^2 \rangle\right) \int_{-\infty}^{\zeta} K(\xi, \xi_1)\, y(t)\, y(\xi_1)\, d\xi_1 \right. \tag{42}$$

$$+\left(\frac{d^2}{d\tau^2}+\langle k^2\rangle\right)\int_{-\infty}^{t}K(t,t_1)\,\langle y(t_1)\,y(\xi)\rangle\,dt_1\Bigg] \qquad (42)\ \text{Cont.}$$

$$+\int_{-\infty}^{t}\int_{-\infty}^{\zeta}\left[\partial(t,t_1,\xi,\xi_1)-K(t,t_1)K(\xi,\xi_1)\right]\langle y(t_1)\,y(\xi_1)\rangle\,dt,\,d\xi,=F(t)\,F(\xi)$$

In the above, $K(t,t_1)$ is the same kernel function as that obtained previously. An analytic prescription for determining the new term, $\partial(t,t_1;\xi,\xi_1)$ is given by the derivation procedure. Again, this prescription is in the form of an infinite series which will be of little direct use in problems of heterogeneous solids. This leads us to again attempt to view Eq. (42) as a phenomenological model.

For most physical problems for which the spring-mass system is illustrative, the analog of Eq. (42) is a complex set of equations with which we have little experience. In the very recent literature on scattering of a radiation field by a random medium, however, some solutions have been presented not only for the analog of Eq. (42) but for the analog of an equation that can be developed by a similar procedure on the four point moment, i.e., $\langle y(t_1)\,y(t_2)\,y(t_3)\,y(t_4)\rangle$. Within the context of the random linearly elastic solid, the general equations on the two point moments defined on the response variables have recently been presented. No solutions of these equations were obtained.

3. STATISTICAL THEORY OF HETEROGENEOUS LINEARLY ELASTIC SOLIDS

3.1 INTRODUCTION

We shall now interpret the conclusions reached in the preceding sections to develop a statistical theory of heterogeneous linearly elastic solids. The theory is intended to be used for making predictions of the response of a variety of materials, including polycrystals, ceramics, a class of composite materials, etc. Each of these materials might be said to possess a disordered substructure.

As noted in the discussion on describing random variables and processes, the concept of an ensemble is inherent to any statistical theory. In the material problem, the ensemble is provided by a collection of materials that emerge from the same fabrication process. A generic interpretation of fabrication process that includes such things as crystal growth, sintering, etc. is intended. Each of the elements of the ensemble is thought to be identical to all of the others in some gross, or macroscopic or average sense. The point by point variations of the material properties will differ for each element of the ensemble due to differences in some uncontrollable parameters of the fabrication process.

In the ensemble approach for describing either the material composition or the response of a specimen made from the material, one no longer seeks

to differentiate between the individual elements of the same fabrication process. The objective, rather, is to differentiate between all of the elements of one fabrication process from all of the elements of a second fabrication process. The most easily visualized method for the lumping together of all the elements of a single fabrication process, which is implied in the statement, is to discuss only averaged quantities, where averaging is taken over the elements. It is when one decides to choose the statistical moments as the averaged quantities to be used in defining the ensemble that the ensemble approach is identified with a probabilistic approach. (See preceding discussion.)

While the logic motivating the idea to consider all specimens that result from a single fabrication process as a single entity is sound, the question can be raised as to what such an approach can predict of a single test specimen. Some might argue that it is the latter problem, i.e., an individual specimen response, that is the physically meaningful one. Two answers can be given. First, the identity of the ensemble approach and the statistical approach allows the interpretation of the averaged response of the ensemble in terms of the expected response in a single test specimen. Further, it is possible to quantify the confidence with which one should view an expected response by making use of higher order information. Secondly, it is possible to interpret the ensemble average of the statistical approach in terms of spatial averages that hold for each and every element of the ensemble in just those cases in which the element invariant spatial averages exist. It is the ergodic hypothesis (see preceding discussion) that allows the interpretation of an ensemble averaged quantity as a prediction to be applied to an individual test specimen.

In principle, ergodicity implies regions of infinite extent. In practice, the requirement is that the stochastic process can be viewed on two length scales. The inner scale, of dimension ℓ, is the one on which the point by point variations of the process are measurable. Any variations in ensemble averages occur over distances that appear unboundedly large when viewed on this scale. Variations in the ensemble averages are measurable on a second scale, termed the outer scale. We let L serve to denote the shortest characteristic dimension of the outer scale. The point by point variations in the stochastic process are too rapid to be perceived when observed on the outer scale. Ergodicity is interpreted as an equation of the ensemble average and a spatial average over a region that appears unboundedly large to an inner scale observer and infinitesimally small to an outer scale observer. We note that the spatial integral could be over one, two or three dimensions depending on the dimensionality of the region within which the two length scales can be identified. In our problems it is usual to think in terms of volume averages. However, in treating bounded solids one can imagine response fields, which, in the vicinity of a bounding surface, are described by stochastic process that are statistically homogeneous in only two dimensions. The two dimensional space is given by the geometry of the bounding surface.

We shall assume that the material properties are described by stochastic processes that are ergodic in the above discussed sense. For the polycrystal, the inner scale is defined by the individual crystals; for a composite, it is defined by the constituent phase geometry. Any finiteness that can be applied to the outer scale arises from the size of the material specimen. As a result of the assumption, a complete material description of the ensemble of specimens can be obtained from measurements taken on any one of the elements of the ensemble.

Except for the most trivial geometries and forcing conditions, the response measures for a solid with statistically homogeneous random variations in material properties will be given by statistically inhomogeneous stochastic processes. If, however, the characteristic linear dimensions that one can assign to the overall geometry of a test specimen and to the forcing conditions are all large relative to the inner scale of the material property variations, then, one can suspect that the stochastic processes needed to describe the response measures can be viewed on two length scales. The inner scale is the inner scale defined by the material property variations. The outer scale is given by the overall dimensions of the test specimen and of the characteristic lengths of the forcing. Assuming the validity of this assumption, a prediction as to the averaged response of the ensemble can be interpreted in terms of the individual response of each element of the ensemble. For specimen geometries and forcing condition that result in response fields that cannot be viewed on two length scales, one can only think of the single specimen test in probabilistic terms.

In discussing the statistical theory of heterogeneous linearly elastic solids, three distinct problems are considered:

1. Mean Field Response of a Statistical Sample of Heterogeneous Linearly Elastic Solids (Statical Loading)

2. Mean Field Response of a Statistical Sample of Heterogeneous Linearly Elastic Solids (Dynamical Loading)

3. The Correlation of Field Quantities in a Statistical Sample of Heterogeneous Linearly Elastic Solids

3.2 MEAN FIELD RESPONSE OF A STATISTICAL SAMPLE OF HETEROGENEOUS LINEARLY ELASTIC SOLIDS (STATICAL LOADING)

In the absence of inertia effects, the equations governing the response of a linearly elastic solid are given by

$$\partial_j \tau_{ij} = F_i \tag{43a}$$

$$\tau_{ij} = C_{ijkl}\,\xi_{kl} \tag{43b}$$

$$\xi_{ij} = 1/2\ (\partial_i u_j + \partial_j u_i) \tag{43c}$$

Here, u_i, ξ_j and τ_{ij} denote the displacement, strain and stress fields respectively; F_i denotes the body force field and C_{ijkl} is the elasticity modul tensor, which gives a material description of the solid. We take C_{ijkl} to be described by statistically homogeneous functions of position. All other inputs to the problem, i.e., the body force field, the location of all bounding surfaces, the conditions to be satisfied on all bounding surfaces, are to be described by ordinary functions.

Application of the method of smoothing leads to the following equations to be satisfied by the ensemble averaged response fields. (The details for these calculations are given in Beran and McCoy (15).)

$$\partial_j \langle\tau_{ij}\rangle = \langle\xi_{ij}\rangle\ 1/2\ (\partial_i\langle u_j\rangle + \partial_j\langle u_i\rangle) \tag{44a}$$

$$\langle\tau_{ij}(x)\rangle = \int C_{ijkl}\ (\underset{\sim}{x},\underset{\sim}{x}_1)\ \langle\xi_{kl}\ (\underset{\sim}{x}_1)\ d\underset{\sim}{x}_1\rangle \tag{44b}$$

$$+\ \phi\ D_{ijkl}\ (\underset{\sim}{x}_1 \underset{\sim}{x}_{i\zeta})\langle\xi_{kl}(\underset{\sim}{x}_{i\zeta})\ \ d\underset{\sim}{x}_{i\zeta} \tag{44c}$$

We use $\int$ and ϕ to denote a volume integral taken over the extent of the specimen and a surface integral taken over all bounding surfaces, respectively. The derivation procedure gives $C_{ijkl}(\underset{\sim}{x},\underset{\sim}{x}_1)$ and $D_{ijkl}(\underset{\sim}{x},\underset{\sim}{x}_i)$ in the form of infinite series, which are of little computational value. In giving a phenomenological interpretation to the equations the prescriptions for the two kernel functions are not used quantitatively. They might be used to give some qualitative insight into the analytic forms, however. For example, the following statements are justified by the first few terms of the infinite series prescriptions. First, $D_{ijkl}(\underset{\sim}{x},\underset{\sim}{x}_i)$ exhibits no singularity as the two points located by $\underset{\sim}{x},\underset{\sim}{x}_1$ approach each other whereas $C_{ijkl}(\underset{\sim}{x},\underset{\sim}{x}_1)$ exhibits a Dirac type singularity as $\underset{\sim}{x} \to \underset{\sim}{x}_1$. As the distance between the two points located by the arguments of C_{ijkl} or D_{ijkl} increases, their values decrease to zero over a length of order ℓ, the maximum correlation length defined by C_{ijkl}. Secondly, the values of the kernel functions depend on the statistics of C_{ijkl}, on the location of the bounding surfaces, and on the type of boundary conditions. Thus, they are not properly interpreted as material parameters. For points far enough removed from all bounding services, the dependence of $C_{ijkl}(\underset{\sim}{x}-\underset{\sim}{x}_i)$ on the boundary condition vanishes. In the case of homogeneous statistics any dependence of $C_{ijkl}(\underset{\sim}{x}-\underset{\sim}{x}_1)$ on absolute position vanishes along with it. In this same region, i.e., points removed from a boundary, we note that $D_{ijkl} \equiv 0$. It can be

speculated that far enough, here, is to be measured relative to the length ℓ.

For specimens with overall dimensions that are very large relative to ℓ, therefore, one may conclude that, for most of the solid, it is valid to take $D_{ijkl} \equiv 0$ and to interpret $C_{ijkl}(\underset{\sim}{x}-\underset{\sim}{x}_1)$ as a material parameter. It is only within thin layers of bounding surfaces that one must consider the surface integral of Eq. (44b) and also must consider the dependence of C_{ijkl} on the boundary conditions that are imposed on the nearby surface and through this dependence, a dependence on absolute position. One might assume that the presence of these layers will have little effect on the mean stress mean strain, and mean displacement fields at points that are far removed from them. Thus, an approximate model for making predictions of the averaged response fields at interior points may be obtained by ignoring these boundary layers allowing Eq. (44b) to be approximated by

$$<\tau_{ij}(x)> = \int C_{ijkl}(\underset{\sim}{x}-\underset{\sim}{x}_1)\ \xi_{kl}\ (\underset{\sim}{x}_1)\ d\underset{\sim}{x}_1 \qquad (45)$$

Only the model that uses Eq. (45) as the relationship between the averaged stress field and the averaged strain field has received any attention. It is to be expected that the general formulation, which uses Eq. (44b) in place of Eq. (45), will receive attention in the future since the region within which the approximate formulation ceases to be valid is very important in predicting failure. Cracks usually are initiated in the vicinity of bounding surfaces and a propagating crack defines its own bounding surface. Since nothing has been reported using the general formulation, however, we are forced to leave it and consider the approximate formulation.

It is almost too easy to identify the form of the approximate formulation with that of a nonlocal elasticity theory. Nonlocal in that it relates the stress at one point in the specimen to the strain at all points. Such theories have often appeared in the early literature. We emphasize that the identification of the equations developed here with those presented elsewhere is a formal mathematical identification. No physical significance should be attached to it without prior justification. Some studies (see below) have already shown that a too hasty interpretation of this mathematical identity in physical terms leads to errors.

The nonlocal formulation may be approximated by a local formulation if the mean response fields vary little over the largest characteristic length associated with the nonlocality, i.e., L. The local formulation is termed an effective modulus formulation. For it, Eq. (45) is approximated by

$$<\tau_{ij}> = C_{ijkl}^{*} <\xi_{kl}> \qquad (46)$$

where

$$C_{ijkl}^{*} = \int C_{ijkl}(\mathcal{R})\, d\mathcal{R} \qquad (47)$$

is termed the effective elasticity moduli tensor. As was noted in dealing with the illustrative example, the conditions for which the effective modulus formulation represents a valid approximation are identical with those that enable our interpreting the ensemble average as a volume average. That the proper equations for predicting the averaged response fields, in the limit of vanishing ℓ/L are those of the classical elasticity theory is consistent with our intuition.

The infinite series prescription for $C_{ijkl}(\mathcal{R})$ that was developed in the derivation procedure could be used in conjunction with Eq. (47) to obtain an analytic prescription for the effective elasticity moduli tensor. Again, because the prescription requires the summation of an infinite series, it is of little computation value. It does, however, provide a clear demonstration that the effective elasticity moduli tensor is dependent on statistical information of all orders. Thus, any attempt to relate the effective properties to only a limited portion of the statistical information, such as knowledge of the percentages of constituent phases, is not logically sound. A statement that one is interested in only approximately determining the effective parameters is meaningless until a consistent manner of estimating the error is provided.

In this report we do not consider, in detail, the literature on making predictions of the effective elasticity moduli tensor based on only partial information of the statistics of C_{ijkl}. A review of this problem from the statistical viewpoint has recently appeared, Beran (17). We do point out, however, that it is possible to place rigorous bounds on the parameters that define the effective material properties. The more information that is put into the bounds, the closer they become. This and the fact that a geometric significance can be attached to different portions of the information put into the bounds has led to a considerable increase in our understanding of the problem For example, we know that volume fraction information of the constituent phases can only be used to shrink the separation distance between bounds on the effective parameters to some minimum. Introducing shape information of the constituent phases into the bounds allows a further decrease in separation between them. Again, however, a minimum separation distance exists if only volume fraction and shape information is to be incorporated. Introducing information of the packing or clustering of constituent phases can lead to a further decrease in separation distance. In Beran (16), some very recent work is discussed that relates volume fraction information to the one point statistical moment, shape information to the three point moments, and clustering information to the four point moments.

An important question to be studied deals with the validity of the solutions of the effective modulus formulation. Obviously, an application

of the formulation that results in solutions that are not consistent with the reasoning that led to the formulation indicates that the idealization is invalid for that particular application. In addition, the discussion of the illustrative problem showed that self-consistency is not sufficient to insure that all is well. Based on that discussion it is wise to suspect that solutions that vary on the ℓ scale will be admitted by the general non-local formulation even if they are not admitted by the effective modulus formulation. A concomitant question to that of the limits of validity of the effective modulus formulation is the question of what one does to extend these limits should such an extension be desirable. An obvious extension that one might try is the expansion and truncation procedure that was introduced in discussing the illustrative problem. This approach is particularly appealing since the resulting formulations contain only differential equations and not integro-differential equations, which are more difficult to solve. Finally, one might note that the formulations that are obtained by the expansion and truncation procedure can be formally identified with equations that arise in the higher order continuum theories, i.e., strain gradient, micropolar, micromorphic, etc., that have received a good deal of attention. This raises the question as to whether or not the identification is purely formal. If physical significance can be attached to the identification, then, it is reasonable to restrict the material parameters that appear in the formulation to conform with the thermodynamic considerations of the higher order continuum theories. Also, it would be reasonable to look to the uniqueness theorems that are provided by these higher order theories to determine the make-up of a properly posed problem.

To answer some of these questions one can look at a specific, although fundamental, problem and investigate the transmission of a force in a medium of infinite extent by the general formulation, by the effective modulus formulation, and by the formulation achieved by an expansion and truncation procedure. To answer the question that pertains to the formality or physical significance of the identity of the formulations that are achieved by the expansion and truncation and those given by the higher order continuum theories one can look to the weakly inhomogeneous solid, for which our model is no longer phenomenological. Thus, we can actually calculate the values of the material parameters in this case. In Beran & McCoy (15) explicit expressions for the material parameters that are contained in the first strain gradient theory were obtained. The results show that the calculated values violate a positive definite requirement on the strain energy density indicating that the identity of the two formulations is only formal. In Beran and McCoy (17) the transmission of a force in a solid of infinite extent was considered with the major conclusions being those termed three in the introduction. In McCoy (18) a uniqueness theorem was presented for the general formulation.

3.3 MEAN FIELD RESPONSE OF A STATISTICAL SAMPLE OF HETEROGENEOUS LINEARLY ELASTIC SOLIDS (Dynamical Loading)

The balance of linear momentum for a continuum subjected to a time harmonic forcing is

$$\partial_j \tau_{ij} + \rho\omega^2 y_j = F_i \tag{48}$$

Where ω denotes the radial frequency and ρ denotes the mass density. We take ρ to be described by a statistically homogeneous function of position. The constitutive relationship and the strain displacement relationship are given by Eqs. (43a) and (43b), respectively.

In McCoy (19) the method of smoothing was applied to the above described equations. The results achieved are extremely complex. Fortunately, this complexity is a consequence of the generality of the general formulation; a generality that is not required in most applications. Of immediate practical interest is the low frequency - long wavelength response of the random linearly elastic solid. In the low frequency-long wavelength limit the general formulation reduces to a dynamical effective modulus theory.

That is,

$$\partial_j \langle\tau_{ij}\rangle + \langle\rho\rangle \omega^2 \langle u_i\rangle + F_i \tag{49a}$$

$$\langle\tau_{i,}\rangle \quad C_{ijkl}^* \langle\xi_{kl}\rangle \tag{49b}$$

$$\langle\xi_{ij}\rangle = 1/2 (\partial_i \langle u_j\rangle + \partial_j \langle u_i\rangle) \tag{49c}$$

where C_{ijkl}^* is identical to that achieved by ignoring inertia effects from the outset. Two important conclusions can be drawn from this set of equations:

1. In the dynamical effective modulus theory it is proper to identify the effective mass density with the average mass density. This is important since it is the average mass density that one obtains by weighing the composite.

2. In the dynamical effective modulus theory it is proper to use the statical effective elasticity moduli tensor.

The effective modulus formulation predicts the conservative, distortionless propagation of a signal. One could question the reasonableness of

this prediction. On physical grounds it would appear to be more reasonable to expect both decay and distortion over long enough propagation distances. This suspicion is further strengthened by the conclusion reached in treating the illustrative problem that the effective parameter formulation does cease to be valid if the observation time is long enough. Further, this illustrative problem showed that corrected formulations achieved by an expansion and truncation procedure were able to make valid predictions of the far field response. In McCoy (19), the expansion and truncation procedure was applied to the general formulation that results from dynamical elasticity equations. The approximate formulation thus achieved does predict both decay and distortion of a propagating signal. The propagation distances that are required for these effects to be observed are of the order of L^4/ℓ^3 and L^3/ℓ^2, respectively. Here, L is a characteristic wavelength of the signal. We note that in a one-dimensional wave-guide the propagation distance required for decay effects to be observed is of the order of L^2/ℓ (20).

The higher-order gradient theories likewise predict dispersive signal propagation. They can also be made to predict decay if one introduces the possibility of energy loss. The energy loss in this case is due to incoherent scattering and not due to the irreversible transformation of energy to a non-mechanical form.

One might also note that the conservative higher-order gradient theories predict solutions that may be termed high frequency - long wavelength solutions. That such solutions are physically meaningful for solids that possess a periodic substructure is well known. One could ask whether such solutions are meaningful for solids that possess a disordered substructure. In McCoy (21), this question was studied for an acoustic medium, i.e., a medium that cannot sustain shear. Here it was shown that such solutions do not exist.

3.4 THE CORRELATION OF FIELD QUANTITIES IN A STATISTICAL SAMPLE OF HETEROGENEOUS LINEARLY ELASTIC SOLIDS

To the author's knowledge the only reported work dealing with the two point moments that can be defined on the response fields for the random linearly elastic solid is a recent paper (McCoy (22)), in which the general formulation analogous to that given for the illustrative problem was presented. None of this is reproduced here because of the complexity of the formulation and because of the preliminary nature of the results obtained.

4. ROLE OF NUMERICAL APPROACHES AND EXPERIMENTATION IN STATISTICAL CONTINUUM THEORIES

There are two inherent difficulties in introducing numerical techniques by means of a Monte Carlo type approach to the study of statistical continuum theories. The first pertains to the very large number of computations that are required by the nature of the problems to be studied. The second pertains to the difficulty in simulating a stochastic process with predetermined statistics on a digital computer. The seemingly endless capacity to improve our computer technology indicates that this first problem will be lessened in the future. Already it is feasible to accomplish, in an acceptable period of time, the numbers of computations required by some statistical continuum problems. The second difficulty, on the other hand, is more permanent.

To illustrate the difficulty we first consider what has been accomplished. Algorithms exist that enable us to construct a set of values that can serve as the ensemble for any random variable. That is, the ensemble can be constructed so as to reproduce any desired probability density function. Such an algorithm may be termed a random number generator. Algorithms do not exist that would enable us to construct a set of pairs of values that can serve as the ensemble for two statistically dependent random variables. It is our inability to build in the constraint required by jointly distributed random variables that prevents our simulating a stochastic process with predetermined statistics.

What might be termed a completely incoherent stochastic process can be simulated by calling on the random number generator for each discreet value(s) defined on the independent variable(s) of the random process. Completely incoherent processes are commonly used in both computer studies and experimental studies requiring a stochastic input. The utility of these studies are limited, however, to problems in which the input process of interest, is completely incoherent or to problems in which the output of interest is independent of any of the statistics of input process measured at two or more points. We note that the completely incoherent process can be introduced as providing an approximation in the elasticity example only in the limiting situation in which all characteristic inner scale dimensions approach zero.

Some degree of statistical dependence for values to be measured at two or more points can be added to the completely incoherent process by transforming it in some specified manner. For example, one might filter a completely incoherent random process to obtain a process that does exhibit some degree of coherence. (The Weiner-Khinchin theorem even enables us to choose the appropriate linear filter to obtain a prescribed correlation function.) Unfortunately, with one notable exception, i.e., passing a Gaussian process through a linear filter, one is rarely able to predict the results of the specified transformation. Our failure to

accomplish this would indeed seem to make the much more difficult problem of predicting the transformation needed to achieve an output with specified statistics intractable.

With this in mind, it is to be expected that much use will be made of Gaussian processes in future numerical studies. One difficulty of using the Gaussian process in the materials problem is that a Gaussian variable can take on any real value. Thermodynamic consideration places value limitations on the material parameters that the process is required to simulate, however. Thus a Gaussian process is not strictly suitable and some modification will be required in the tails of the distribution functions.

As an alternate to using the Gaussian process, it is to be expected that attempts will be made to simulate fabrication processes in the computer. As an example, we consider the following process that is described in Beran (16) for generating a two-dimensional random function that may be representative of some two phase materials. We first choose N points per unit volume at random in a plane surface. These points are termed Poisson points. Next we choose, at random, $V_1 = M/N$ of the Poisson points to belong to one material and $1-V_1$, points to belong to the second material. Finally, we decide which material may be associated with all other points by stating that the material at a point is the same as the material at the nearest Poisson point. Once the stochastic function has been generated as described, we can measure the resulting statistics.

Again, one might simply read in results of physical experiments that define correlated stochastic processes. The statistics of these processes like those of our two phase Poisson material can now be measured. The objective here is to build up a library of stochastic processes from which we can select one that has the desired characteristics for a given study.

Computer studies of statistical continuum problems are all relatively recent. Deane (23) considers some randomly spaced defects in a periodic lattice. (See also Adams and Tsai (24).) For the more general problem, reference can be made to Bradley and Herrmann (25) for numerical studies of propagation through a turbulent medium. Two research programs are currently being accomplished at Catholic University which involve numerical solutions of statistical continuum problems. One considers the scattering of acoustic waves by a randomly rough surface and the second considers the propagation of signals through a one dimensional random waveguide.

The ultimate test of the validity and the utility of the theoretical ideas discussed in this paper will be experimental tests. Thus, experimentation will play a central role in all aspects of the problem. It is therefore unfortunate, that so little experimental work that recognizes the inherently statistical nature of the problem has been reported. Indeed, of all of the ideas discussed, it is only the problem of predicting the

effective properties that has received any attention at all. And here, only the experiments of Corson (26) considered other than volume fraction information to describe the substructure geometry. Clearly, a great deal more experimentation is indicated.

5. FUTURE STUDIES

Most of the work reported in this paper could rightly be classified as laying a theoretical foundation for a statistical interpretation of the analysis of solids with disordered substructures. While additional work along this same line is warranted, the number of results already achieved is great enough to warrant a slight shift in emphasis in our future efforts. In the immediate future some emphasis should be placed on validating, by both numerical and experimental procedures, the results obtained to date. In addition, some emphasis should be placed on demonstrating the physical importance of the results obtained to date. Listed below are some areas of future studies that incorporate this shift in emphasis.

Problems in which the mean field quantities are the quantities of interest may be classified by the validity or the lack of validity of making predictions based on the effective modulus formulation. For the former class of problems, the formulation of any problem is well understood as is the interpretation of the predictions to be made. For this class, an ergodic hypothesis is valid and all averages may be viewed as spatial averages. Thus, the only remaining question of interest is the relationship between the effective properties of the formulation and measures of inner scale point by point variations in material properties. The statistical interpretation has contributed two significant results to this latter problem. First, it has shown that volume fraction information of the constituent phases does not alone determine the effective properties. Shape information, packing information, etc. are also important. Secondly, the statistical interpretation provides a hierarchy of measures for incorporating this added information. A final step is the need to relate the measures in the hierarchy to parameters in a fabrication process. While shape information is readily related to a fabrication process parameter, packing information is not. One might imagine that the size distribution on constituent phases enters the picture but the relationship between the two is not completely clear. Once the final step is achieved, the tools will be at hand to synthesize materials with desired effective properties by changing the substructure geometry. It seems clear that a task that should be accomplished in order to carry out this program is to develop the capability of measuring the hierarchy of statistical measures. It might be parenthetically remarked that this capability holds promise of being useful in a wide range of problems.

In considering situations in which the effective modulus formulation predicts results that are at variance with physical reality it is convenient

to differentiate between two different types of errors. One type of error to be expected is characterized by its local nature. Any sharp discontinuity in specimen loading or specimen geometry is to be expected to give rise to a stress field in the immediate vicinity of the discontinuity that will differ significantly from that predicted by an effective modulus formulation. It is important, therefore, to use a generalized theory for making predictions for a class of fracture mechanics problems. It has been demonstrated that the strain gradient theories are not to be expected to represent a valid generalized theory. What is needed is a theory that incorporates the non-local nature of the integral operator that relates the averaged stress and strain fields. To make this type theory practical will require our limiting the forms of the kernel function. As discussed in the report, limitations on the forms can be achieved on several fronts. Ultimately, however, the author expects that limitations that result from our intuition in conjunction with a suitable test program will provide the key to developing a theory that will have computational utility. A fruitful area of study would be to devise experimental tests in which the effective modulus theory is to be expected to be invalid. For these tests one can then study the ability of a restricted form of the general formulation; obtained by postulating a form of the kernel function; to reproduce the test data.

A second type error to be expected will grow with observation time or observation distance. Signal distortion and signal decay, which are not predicted by the dynamical effective modulus theory, increase with propagation distance. The importance, for a vibrating system, of the irreversible transformation of mechanical energy to an incoherent form similarly grows with observation time. It has been demonstrated that the field equations of a dynamical strain gradient formulation can be made to reproduce those features of the exact infinite medium dispersion spectrum that are relevant to these problems provided one correctly chooses the material parameters of the formulation. However, to show that a dynamical strain gradient formulation is a proper generalized formulation for reproducing long time solutions requires one to look at the complete formulation. That is, one must look at the auxiliary conditions that are to be specified for a well-posed problem as well as the nature of the solutions admitted by the field equations. The author suspects that here difficulties will be encountered by a dynamical strain gradient formulation. Further study is required into a proper computationally useful formulation for reproducing long time solutions of a dynamical system. Since the ultimate test of this formulation will be a laboratory test, it is necessary to develop the capability of making measurements over a sufficiently long observation time or observation distance that will not be completely submerged in the "noise" field that results from the multiple interaction of the strain signals and the bounding surfaces of the specimen.

It is clear, for example, that since the strength characteristics of the matrix of a fiber reinforced composite differs substantially from the strength characteristics of the fiber, it is not reasonable to attempt to relate failure to an averaged value of the stress taken over both the matrix and the fiber phases. Some information on the variance of the stress

about this average is of obvious practical importance. It is for this reason that a formulation was developed to provide predictions of the correlation functions defined on the stress field. Although this formulation is extremely complex, it is important to begin studies that will enable our obtaining some simple solutions to gain further insight into the problem.

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